International Journal of Theoretical Physics, Vol. 36, No. 3, 1997

Effects of a Small Deviation from Fermi Statistics

C. Wolf[1]

Received February 16, 1996

By considering a generalized statistics with occupation numbers between Bose–Einstein and Fermi–Dirac statistics we study the resultant distribution when the states differ by a small factor from a Fermi–Dirac distribution. Both the Fermi energy and any level crossing phenomena are sensitive to such statistics; in particular, the electrical conductivity and the free electron heat capacity of fermions at low temperatures receive corrections due to alterations of Fermi–Dirac statistics.

1. INTRODUCTION

It is fair to say that of all the principles of quantum theory, the exclusion principle and the existence of Bose–Einstein (BE) and Fermi–Dirac (FD) statistics represent the most mysterious and ad hoc fundamental axioms of the theory (Lamoreaux, 1992; Dyson, 1967). Yet so much depends on them; the existence of atomic stability, nuclear structure, the hadron spectrum, and the very existence of the color degree of freedom represent some of the consequences of FD statistics (Okun, 1989). One might wonder whether the exclusion principle is a consequence of an underlying symmetry of space-time or is a result of topological considerations in spin space (Geroch and Horowitz, 1979). This latter idea emerges from the fact that fermions feel each other's presence even without an interaction through a potential. In fact, until the study on the (2 + 1)-dimensional quantized Hall effect (Girven and Prange, 1987) along with the anomalous statistics generated by (2 + 1)-dimensional anayons (Luscher, 1989; Semmenoff and Sodamo, 1989) there was virtually no good reason to doubt the FD and BE statistics. Recently, Wu (1994), motivated by the work of Haldane (1991), discussed a generalized statistics which interpolates between BE and FD statistics. In a previous

[1]Department of Physics, North Adams State College, North Adams, Massachusetts 01247.

0020-7748/97/0300-0625$12.50/0

paper we used the entropy for such a generalized statistics to derive a formula for the distribution function for particles obeying such statistics (Wolf, 1995). When we consider a small deviation from BE statistics we can discuss the spectral distribution of anomalous photons obeying such a distribution. The spectral distribution differs from the Rayleigh–Jeans law at low frequencies, which would effect the cosmic microwave background if such anomalous photons are in equilibrium with normal photons. In the present paper we study the case where there is a small deviation from FD statistics. After we derive the distribution law we point out that anomalies would show up in the Fermi energy and transport phenomena such as electrical conductivity at extremely low temperatures. We also calculate anomalous contributions to the free electron specific heat at low temperatures.

2. SMALL DEVIATIONS FROM FD STATISTICS

We begin by giving a brief derivation of the formula for the distribution of particles in a system admitting anomalous statistics; according to Wu (1994), we write the number of ways of realizing N_i particles in g_i cells as

$$W_i = \frac{[g_i + (N_i - 1)(1 - \alpha)]!}{N_i!\,(g_i - \alpha N_i - (1 - \alpha)]!} \tag{2.1}$$

Here for $\alpha = 1$ (Fermi–Dirac case) we have

$$W_i = \frac{g_i!}{N_i!\,(g_i - N_i)!}$$

for $\alpha = 0$ (Bose–Einstein case) we have

$$W_i = \frac{(g_i + N_i - 1)!}{N_i!\,(g_i - 1)!}$$

For $0 < \alpha < 1$ we have for the total entropy

$$S = k \ln_e \prod_i \frac{(g_i + (N_i - 1)(1 - \alpha))!}{N_i!\,(g_i - \alpha N_i - (1 - \alpha))!} \tag{2.2}$$

Taking the natural log of equation (2.2) and varying with respect to N_i, we have for equation (2.2) upon summing (here we maximize S)

$$\sum_i \begin{pmatrix} (1 - \alpha) \ln_e(g_i + (N_i - 1)(1 - \alpha)) + (1 - \alpha) - (1 - \alpha) \\ -\ln_e N_i + 1 - 1 + \alpha \ln_e(g_i - \alpha N_i - (1 - \alpha)) + \alpha - \alpha \end{pmatrix} dN_i = 0 \tag{2.3}$$

In equation (2.3) we set $N_i - 1 \approx N_i$ and neglect $(1 - \alpha)$ in comparison to $g_i - \alpha N_i$; this gives

$$\sum_i \left(\ln_e \frac{(g_i + N_i(1 - \alpha))^{1-\alpha}(g_i - \alpha N_i)^\alpha}{N_i} \right) \delta N_i = 0 \tag{2.4}$$

We next employ the constraints for the number of particles and total energy

$$\sum \delta N_i = 0 \tag{2.5}$$

$$\sum \epsilon_i \delta N_i = 0 \tag{2.6}$$

For the Lagrange multipliers we have μ/τ and $-1/\tau$ for equations (2.5) and (2.6), respectively. This gives in combination with equation (2.4)

$$\sum_i \left(\ln_e \frac{(g_i + N_i(1 - \alpha))^{1-\alpha}(g_i - \alpha N_i)^\alpha}{N_i} + \frac{\mu}{\tau} - \frac{\epsilon_i}{\tau} \right) dN_i = 0$$

Setting the coefficient of dN_i equal to 0 gives

$$\frac{(g_i + (1 - \alpha)N_i)^{1-\alpha}(g_i - \alpha N_i)^\alpha}{N_i} = e^{(\epsilon_i - \mu)/\tau} \tag{2.7}$$

Here μ is the chemical potential, $\tau = k_B T$ is the normalized temperature, ϵ_i is the energy of level i, and N_i is the occupation of level i.

In equation (2.7) we let $\alpha = 1 - \epsilon_1$ (ϵ_1 is a small parameter). Equation (2.7) becomes after taking the natural log of both sides

$$\epsilon_1 \ln_e g_i \left(1 + \frac{\epsilon_1 N_i}{g_i} \right) + (1 - \epsilon_1) \ln_e (g_i - N_i) \left(1 + \frac{\epsilon_1 N_i}{g_i - N_i} \right) - \ln_e N_i = \frac{\epsilon_i - \mu}{\tau}$$

Expanding the above equation, we have, upon keeping terms to first order in ϵ_1,

$$\epsilon_1 \ln_e g_i + (1 - \epsilon_1) \ln_e (g_i - N_i) + \frac{\epsilon_1 N_i}{(g_i - N_i)} - \ln_e N_i = \frac{\epsilon_i - \mu}{\tau}$$

or

$$\ln_e \frac{g_i - N_i}{N_i} = \epsilon_1 \ln_e \frac{g_i - N_i}{g_i} - \frac{\epsilon_1 N_i}{g_i - N_i} + \frac{\epsilon_i - \mu}{\tau} \tag{2.8}$$

or

$$\frac{g_i - N_i}{N_i} = e^{(\epsilon_i - \mu)/\tau} \left(1 - \frac{\epsilon_1 N_i}{g_i - N_i} \right) \tag{2.9}$$

Here we approximate

$$\left(\frac{g_i - N_i}{g_i}\right)^{\epsilon_1} \approx 1 \qquad (\epsilon_1 \text{ small})$$

Solving equation (2.9) for N_i to first order in ϵ_1 gives

$$N_i \simeq \frac{g_i(1 + \epsilon_1)}{e^{(\epsilon_i - \mu)/\tau} + 1} - \frac{g_i e^{(\epsilon_i - \mu)/\tau}\epsilon_1}{(e^{(\epsilon_i-\mu)/\tau} + 1)^2} \tag{2.10}$$

For $\tau \to 0$, $\epsilon < \mu$,

$$N_i \simeq g_i(1 + \epsilon_1)$$

Thus, if

$$g_i = \frac{8\pi P^2 dP}{h^3} V$$

we have

$$N = (1 + \epsilon_1)\frac{8\pi P_M^3}{3h^3} V$$

where V is the spatial volume and P_M is the maximum momentum. Thus

$$P_M = \left(\frac{3Nh^3}{(1 + \epsilon_1)8\pi V}\right)^{1/3}$$

and

$$\epsilon_F = \frac{1}{2m}\left(\frac{3Nh^3}{(1 + \epsilon_1)8\pi V}\right)^{2/3}$$

for the Fermi energy.

Thus the Fermi energy is diminished for $\epsilon_1 \neq 0$.

Equation (2.10) in velocity space reads for unit spatial volume and no potential (for spin up and spin down)

$$dN = \left[\frac{2(1 + \epsilon_1)m^3}{(e^{(\epsilon-\mu)/\tau} + 1)} - \frac{2\epsilon_1 m^3 e^{(\epsilon-\mu)/\tau}}{(1 + e^{(\epsilon-\mu)/\tau})^2}\right]\frac{dV_x dV_y dV_z}{h^3}$$

$[\epsilon_1 = \epsilon(V_x, V_y, V_z)]$, or for the distribution function

$$f_0 = \left[\frac{2(1 + \epsilon_1)m^3}{e^{(\epsilon-\mu)/\tau} + 1} - \frac{2\epsilon_1 m^3 e^{(\epsilon-\mu)/\tau}}{(1 + e^{(\epsilon-\mu)/\tau})^2}\right]\frac{1}{h^3} \tag{2.11}$$

where $\epsilon = \frac{1}{2}m(V_x^2 + V_y^2 + V_z^2)$.

For the Boltzmann equation in an x-component electric field we have

$$f = f_0 - \frac{\mathscr{E}e}{m}\tau_r \frac{\partial f_0}{\partial V_x} \tag{2.12}$$

($\mathscr{E}$ is the electric field) to first order in τ_r (relaxation time). The electric current is

$$J_x = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} -\frac{\mathscr{E}e^2}{m}\tau_r V_x \frac{\partial f_0}{\partial V_x}\, dV_x\, dV_y\, dV_z \tag{2.13}$$

Since f_0 depends on ϵ_1 as in equation (2.11), the electric current would depend on ϵ_1 through equation (2.13).

3. CORRECTIONS TO THE ELECTRICAL CONDUCTIVITY AND FREE ELECTRON SPECIFIC HEAT OF ELECTRONS AT LOW TEMPERATURE DUE TO ANOMALOUS STATISTICS

If we substitute equation (2.11) into equation (2.13), we find

$$J_x = -2e^2\mathscr{E}\left(\frac{m}{h}\right)^3\tau_r \times \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} V_x^2 \frac{\partial}{\partial\epsilon}\left(\frac{1+\epsilon_1}{e^{(\epsilon-\mu)/\tau}+1} - \frac{\epsilon_1 e^{(\epsilon-\mu)/\tau}}{(e^{(\epsilon-\mu)/\tau}+1)^2}\right) dV_x\, dV_y\, dV_z \tag{3.1}$$

For low τ, $\epsilon_F = \mu >> \tau$ (Kittel, 1958)

$$\frac{\partial}{\partial\epsilon}\left(\frac{1}{e^{[(\epsilon-\epsilon_F)/\tau]}+1}\right) = -\delta(\epsilon-\epsilon_F)$$

($\epsilon_F \approx \mu$ = Fermi energy).

Equation (3.1) becomes

$$J_x = 2e^2\mathscr{E}\left(\frac{m}{h}\right)^3\tau_r \iiint V_x^2\left[(1+\epsilon_1)\delta(\epsilon-\epsilon_F) - \frac{2\epsilon_1 e^{(\epsilon-\epsilon_F)/\tau}\delta(\epsilon-\epsilon_F)}{e^{(\epsilon-\epsilon_F)/\tau}+1} + \epsilon_1\delta(\epsilon-\epsilon_F)\right] dV_x\, dV_y\, dV_z \tag{3.2}$$

Evaluating equation (3.2) gives (Kittel, 1958)

$$J_x = e^2\mathscr{E}\tau_r\epsilon_F^{3/2}\left(\frac{16\pi}{3}\right)\frac{\sqrt{2}}{m^{5/2}}\left(\frac{m}{h}\right)^3(1-\epsilon_1) \tag{3.3}$$

where

$$\epsilon_F = \frac{1}{2m}\left(\frac{3Nh^3}{(1+\epsilon_1)8\pi V}\right)^{2/3}$$

Combining equation (3.3) and the above modified expression for ϵ_F gives

$$J_x = e^2\mathscr{E}\tau_r(\epsilon_{F_0})^{3/2}\,\frac{16\pi}{3}\,\frac{(\sqrt{2})}{m^{5/2}}\left(\frac{m}{h}\right)^3(1-2\epsilon_1) \tag{3.4}$$

$$J_x = J_{0_x}(1-2\epsilon_1)$$

Here ϵ_{F_0} is the Fermi energy without anomalous statistics.

We now calculate the corrections to the free electron specific heat induced by the anomalous statistics. Following the development of Kittel (1969), we have for the energy of a free electron gas above the value at $T = 0$ (M is the mass of the electron, L^3 is the volume)

$$\Lambda U(T) = \int_0^\infty \left(\frac{1+\epsilon_1}{(e^{(\epsilon-\epsilon_F)/\tau}+1)} - \frac{\epsilon_1 e^{(\epsilon_1-\epsilon_F)/\tau}}{(1+e^{(\epsilon-\epsilon_F)/\tau})^2}\right)\frac{L^3}{2\pi^2}\left(\frac{2M}{\hbar}\right)^{3/2}\epsilon^{3/2}\,d\epsilon$$
$$-\int_0^{\epsilon_F}(1+\epsilon_1)\,\frac{L^3}{2\pi^2}\left(\frac{2M}{\hbar^2}\right)^{3/2}\epsilon^{3/2}\,d\epsilon \tag{3.5}$$

Here

$$\frac{L^3}{2\pi^2}\left(\frac{2M}{\hbar^2}\right)^{3/2}\epsilon^{1/2}d\epsilon$$

is the number of electron states (spin up and spin down) between ϵ and $\epsilon + d\epsilon$ in the spatial volume $V = L^3$.

In equation (3.5)

$$f(\epsilon) = \frac{1+\epsilon_1}{e^{(\epsilon-\epsilon_F)/\tau}+1} - \frac{\epsilon_1 e^{(\epsilon-\epsilon_F)/\tau}}{(e^{(\epsilon-\epsilon_F)/\tau}+1)^2} \qquad [\epsilon_F \approx \mu(0)] \tag{3.6}$$

Upon using the relation for the equality of the number of particles at $\tau = 0$ and $\tau = \tau$,

$$\epsilon_F\int_0^{\epsilon_F} f(\epsilon)D(\epsilon)\,d\epsilon + \epsilon_F\int_{\epsilon_F}^\infty f(\epsilon)D(\epsilon)\,d\epsilon$$
$$= \epsilon_F\int_0^{\epsilon_F}(1+\epsilon_1)D(\epsilon)\,d\epsilon \tag{3.7}$$

[$D(\epsilon)$ is the density of states in ϵ space] we may rewrite equation (3.5) as

$$\Delta U = \int_{\epsilon_F}^{\infty} (\epsilon - \epsilon_F) f(\epsilon) D(\epsilon)\, d\epsilon + \int_0^{\epsilon_F} d\epsilon\, (\epsilon - \epsilon_F)[1 + \epsilon_1 - f(\epsilon)] D(\epsilon) \tag{3.8}$$

For the specific heat of the free electron gas we have

$$C_v = \frac{d(\Delta U)}{dT} = \int_0^{\infty} (\epsilon - \epsilon_F) \frac{df(\epsilon)}{dT} D(\epsilon)\, d\epsilon$$

$$\simeq D(\epsilon_F) \int_0^{\infty} (\epsilon - \epsilon_F) \frac{df(\epsilon)}{dT}\, d\epsilon \tag{3.9}$$

Writing $(\epsilon - \epsilon_F)/kT = x$ and approximating the integral in equation (3.9) from $-\infty$ to $+\infty$ in x, we have, upon carrying out the derivatives in equation (3.6),

$$C_v = k_B^2 T D(\epsilon_F)\left[(1 + 2\epsilon_1) \int_{-\infty}^{\infty} \frac{x^2 e^x\, dx}{(e^x + 1)^2} - 2\epsilon_1 \int_{-\infty}^{\infty} \frac{x^2 e^{2x}}{(e^x + 1)^3}\right] \tag{3.10}$$

where

$$D(\epsilon_F) = \frac{V}{2\pi^2}\left(\frac{2M}{\hbar^2}\right)^{3/2} \epsilon_F^{1/2}$$

$$\simeq \frac{V}{2\pi^2}\left(\frac{2M}{\hbar^2}\right)^{3/2} \epsilon_{F0}^{1/2}\left(1 - \frac{\epsilon_1}{3}\right) = D(\epsilon_{F0})\left(1 - \frac{\epsilon_1}{3}\right)$$

after using the corrected value of ϵ_F (ϵ_{F_0} is the Fermi energy in the absence of anomalous statistics).

Equation (3.10) becomes

$$C_v \simeq k_B^2 T D(\epsilon_{F0})\left(1 - \frac{\epsilon_1}{3}\right)\left[\frac{\pi^2}{3} + 2\epsilon_1 \int_{-\infty}^{\infty} \frac{x^2 e^x}{(e^x + 1)^3}\, dx\right]$$

$$\simeq k_B T D(\epsilon_{F0})\left[\frac{\pi^2}{3} + \epsilon_1\left(2K_0 - \frac{\pi^2}{9}\right)\right] \tag{3.11}$$

where $K_0 \simeq 2/27$. We thus have a negative correction term to the specific heat proportionatal to ϵ_1. In equations (3.4) and (3.11) we see that the anomalous statistics decreases the free electron specific heat and decreases the free electron electric current. The problem is to separate out corrections due to variations of the relaxation time with energy in the electric current from anomalous statistics corrections and also separate out electron phonon corrections to C_v and spin fluctuation corrections to C_v from anomalous corrections to the specific heat. Also, corrections due to the fact that the number of free electrons may not be equal to the number of ions would make an estimate of the anomalous parameter a very delicate matter.

4. EXPERIMENTAL PROBES OF ANOMALOUS STATISTICS

From equations (3.4) and (3.11) we have seen that the free electron current and the specific heat are diminished due to the anomalous statistics in proportion to ϵ_1. The essential reason for this is that more electrons can be put in lower energy levels, thus producing a diminished Fermi energy and a diminished free electron total energy and free electron current. Since the relaxation time τ_r in equation (3.4) can be dependent on the energy, it would be difficult to separate out corrections due to the energy-dependent relaxation time from corrections due to anomalous statistics. Also, for the specific heat of metals at low temperatures the uncertainty in the number of free electrons as mentioned above would lead to uncertainties in the specific heat that would compete with corrections due to anomalous statistics. To set limits on ϵ_1, however, we may note that certain elements have a lower experimental value of the linear term in the free electron specific heat than the theoretical value. For instance, for BE (Ashcroft and Mermin, 1976),

$$(C_v)_{\text{TH}} = 1.2 \times 10^{-4}T \quad \frac{\text{cal}}{\text{mol K}}$$

$$(C_v)_{\text{EX}} = 0.5 \times 10^{-4}T \quad \frac{\text{cal}}{\text{mol K}}$$

If we attribute 10% of the discrepancy to anomalous states from equation (3.11), we have

$$\left(\frac{3}{\pi^2}\right)(1.2 \times 10^{-4})\epsilon_1\left(\frac{\pi^2}{9} - \frac{4}{27}\right) \simeq (0.7 \times 10^{-4})(10^{-1})$$

$$\epsilon_1 \simeq 0.2$$

for the anomalous parameter. For other metals the experimental values are larger than those predicted by equation (3.10). The problem is that for most metals and alloys electron–phonon contributions to C_v and spin fluctuation contributions to C_v (Wire *et al.*, 1983) at low T are of the same order of magnitude as or greater than the free electron contribution, thus making an estimate of ϵ_1 difficult. If we turn to superconductivity to try to establish limits in ϵ_1, we find that the critical temperature, the energy gap, the critical field (to destroy superconductivity), and the ratio of the specific heat of the superconducting to normal metal at the critical temperature may be sensitive to anomalous electron statistics (Phillips, 1959). However, all these quantities when measured experimentally are within 10% of the theoretical predictions (Roberts, 1964), thus making it difficult to set limits on ϵ_1, the anomalous parameter. Also in the BCS theory (Bardeen *et al.*, 1957) the neglect of the band structure and the estimation of an effective potential introduce differences between the experimental and theoretical predictions of the critical temperature, energy gap, and other characteristic features of the superconducting state. The best way to put limits on the anomalous parameter ϵ_1 is to pick a specific metal where the electron–phonon, and spin-fluctuation, and other corrections are well known and then make accurate measurements on the specific heat to ascertain limits on ϵ_1.

5. CONCLUSION

The above discussion of anomalous fermion statistics has suggested that both the free electron current and free electron specific heat are sensitive to anomalous statistics. The difficulty in setting limits on ϵ stems from the uncertainty in the relaxation time (for the free electron current) and the electron phonon and spin fluctuation contributions to C_v (for the free electron specific heat). However, the issues raised in this paper concerning the existence of anomalous statistics suggest experimental "windows" through which such anomalies might be observed. This is the spirit of this investigation. Certainly the discovery of anomalous statistics for fermions would have far-ranging consequences in both condensed matter physics and elementary particle physics.

ACKNOWLEDGMENTS

I like to thank the Physics Departments at Williams College and Harvard University for the use of their facilities.

REFERENCES

Ashcroft, N. W., and Mermin, N. D. (1976). *Solid State Physics,* Holt, Reinhart and Winston, New York, p. 48.

Bardeen, J., Cooper, L. N., and Schrieffer, J. R. (1957). *Physical Review,* **108**, 1175.

Dyson, F. J. (1967). *Journal of Mathematical Physics,* **8**, 1538.

Geroch, R., and Horowitz, G. T. (1979). The global structure of spacetime, in *Einstein Centenary Survey,* S. W. Hawking and W. Israel, eds., Cambridge University Press, Cambridge, p. 217.

Girven, S., and Prange, R. (1987). *The Quantum Hall Effect*, Springer-Verlag, Berlin.
Haldane, F. D. M. (1991). *Physical Review Letters*, **67**, 937.
Kittel, C. (1958). *Elementary Statistical Physics*, Wiley, New York, p. 198.
Kittel, C. (1969). *Thermal Physics*, Wiley, New York, p. 231.
Lamoreaux, S. K. (1992). *International Journal of Modern Physics*, **7**, 6691.
Luscher, M. (1989). *Nuclear Physics B*, **326**, 557.
Okun, L. B. (1989). *Comments on Nuclear and Particle Physics*, **XII**(3), 99.
Phillips, N. E. (1959). *Physical Review*, **114**, 676.
Roberts, B. W. (1964). *Progress in Cryogenics*, **4**, 161.
Semmenoff, G. W., and Sodamo, R. (1989). *Nuclear Physics B*, **328**, 753.
Wire, M. S., Stewart, G. R., Johanson, W. R., Fisk, Z., and Smith, J. L. (1983). *Physical Review B*, **23**, 6518.
Wolf, C. (1995). *Nuovo Cimento B*, **110**, 1481.
Wu, Y. S. (1994). *Physical Review Letters*, **73**, 922.